proof: Exercise. Hint: use the $\tau \dashv N$ adjunction and consider the maps $\tau(\Lambda^n_k) \longrightarrow \tau(\Delta^n)$ for $0 < k < n$.

$\square$

Collections of morphisms of the form ${}^{\boxed{\diagup}}S$ (or $S^{\boxed{\diagup}}$) have some remarkable "closure" properties.

def 5: Let $S \subseteq \mathrm{Mor}(\mathcal{C})$. We say that:

* S is closed under pushouts if for every pushout diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\ulcorner" very near end] & A' \arrow[d, "\gamma'"] \\ B \arrow[r] & B' \end{tikzcd}$$

, $\gamma \in S \Rightarrow \gamma' \in S$.

* S is closed under retracts if for every retract diagram in $\mathcal{C}$:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "id"] & A' \arrow[r] \arrow[d, "f'"'] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[rr, bend right, "id"'] & B' \arrow[r] & B
\end{tikzcd}$$

(expressing that f' is a retract of f)

we have $f \in S \Rightarrow f' \in S$.

* S is closed under transfinite compositions if for all ordinals α and a functor $F : [\alpha]^{=\alpha \cup \{\alpha\}} \longrightarrow \mathcal{C}$ with $\forall\ 0 < \lambda \leq \alpha$ with λ limit ordinal,

$$F(\lambda) \simeq \operatorname*{colim}_{\gamma < \lambda} F(\gamma)$$

$$\Big(\text{For all } \beta < \alpha,\ F(\beta < \beta+1) \in S \Big) \Rightarrow F(0 < \alpha) \in S$$

[ex For $\alpha = \omega = (\mathbb{N}, \leq)$, then $[\omega] = \omega + 1 = \mathbb{N} \cup \{\infty\}$

and the condition says that if

$$F(\infty) \simeq \operatorname*{colim}_{n \in \mathbb{N}} \left(F(0) \underset{\in S}{\to} F(1) \underset{\in S}{\to} F(2) \underset{\in S}{\to} \cdots \right)$$

then $F(0) \longrightarrow F(\infty)$ is in S.]

* S is (weakly) saturated if it is stable under pushouts, retracts and transfinite compositions.

Rmk: • There is a related notion of "strongly" saturated class which plays a role in the theory of localizations of (∞-) categories, hence the terminology.

• The transfinite composition for $\alpha = 0$ (resp. $\alpha = 2$) means that S contains all isos (resp. is stable by compositions).

Lemma 6: Let C admit arbitrary coproducts. Then any saturated collection in C is stable under coproducts.

proof: Let S be a saturated collection, and $(f_\alpha : A_\alpha \to B_\alpha)$ be a set of morphisms such that the coproduct

$$\coprod f_\alpha : \coprod A_\alpha \to \coprod B_\alpha \text{ exists.}$$

We can construct $\coprod f_\alpha$ as a transfinite composition of pushouts. Ex:

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d] & B_0 \arrow[r, "id"] \arrow[d] & B_0 \arrow[d] \\
A_0 \amalg A_1 \arrow[r] & A_1 \amalg B_0 \arrow[r] & B_0 \amalg B_1 \\
\vdots & & \vdots \\
\coprod_{n \in \mathbb{N}} A_n \arrow[rr, "\coprod f_\alpha"] & & \coprod_{n \in \mathbb{N}} B_n
\end{tikzcd}$$

$\square$

This notion seems very artificial at first, but as we will see, it is in fact quite natural in view of Quillen's "small object argument".

First, we have:

prop 7: Let $S \subseteq \mathrm{Mor}(\mathcal{C})$ be any collection of morphisms. Then ${}^{\boxslash}S$ is saturated.

proof:

Stable under pushouts:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "f"', "{}^{\boxslash}S \ni"] & A' \arrow[r] \arrow[d] & X \arrow[d, "\in S"] \\
B \arrow[r] \arrow[urr, dashed] & B' \arrow[r] \arrow[ur, dashed] \arrow[ul, phantom, "\ulcorner", very near start] & Y
\end{tikzcd}
$$

where $\longrightarrow$ exists because $f \in {}^{\boxslash}S$

$\dashrightarrow$ exists because of universal property of the pushout.

Stable under retracts:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "{}^{\boxslash}S \ni g"'] & A' \arrow[r] \arrow[d, "g'"'] \arrow[rr, bend left] & A \arrow[d, "g"] & X \arrow[d, "\in S"] \\
B \arrow[r] \arrow[urrr, dashed] & B' \arrow[r] \arrow[urr, dashed] \arrow[rr, bend right] & B \arrow[ur, dashed] & Y
\end{tikzcd}$$

where the two (equal) $\rightarrow$ exist because of $g \in {}^{\boxslash}S$, the $\rightarrow$ is just defined as composition, and everything commutes because of the retract property $\xrightarrow{id}$, $\xrightarrow{id}$.

Stable under transfinite compositions:

For notational simplicity I explain the case $\omega = (\mathbb{N}, \leq)$, the general argument is the same:

$$
\begin{tikzcd}
F(0) \arrow[r] \arrow[d, "{}^{\boxslash}S \ni"'] & X \arrow[dddd, "\in S"] \\
F(1) \arrow[ur, dashed] \arrow[d, "{}^{\boxslash}S \ni"'] & \\
F(2) \arrow[uur, dashed] \arrow[d, dashed, no head] & \\
\vdots & \\
F(\infty) \arrow[r] \arrow[uuuur, dashed] & Y
\end{tikzcd}
$$

where:
- $\rightarrow$'s are constructed by induction using that $F(i < i+1) \in {}^{\boxslash}S$.
- $\rightarrow$ is constructed from the universal property of colimits. $\square$

An intersection of saturated collections is saturated. This implies the following is well-defined:

def 8: Let C be a category and S_0 be a collection of morphisms in C. There is a smallest saturated collection $\overline{S_0}$ which contains S_0. We call it the saturated collection generated by S_0.

As a first example, we have:

prop 9: The collection of all monomorphisms in $sSet$ is the saturated collection generated by $(\partial\Delta^n \hookrightarrow \Delta^n)_{n\geq 0}$.

proof: The fact that (monos) is saturated is easy and left as an exercise.

$$\Rightarrow \overline{(\partial\Delta^n \hookrightarrow \Delta^n)} \subseteq (\text{monos}).$$

· The fact that $(\text{monos}) \subseteq \overline{(\partial\Delta^n \hookrightarrow \Delta^n)}$ follows from the existence of the skeletal filtration (Prop I.24) for any monomorphism. $\square$

Cor 10: 1) Let $g : X \to Y$ in sSet. TFAE

a) g is a trivial fibration $\left(\text{i.e.} \begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \right)$

b) g has the right lifting property wrt monos.

2) Let $g : X \to Y$ be a trivial Kan fibration.

Then g admits a section $s : Y \to X$.

Moreover, the composition $s \circ g : X \to X$ is homotopic to id_X over Y: there exists

$$\begin{tikzcd} X \times \Delta^1 \arrow[rr, "h"] \arrow[dr] & \equiv & X \arrow[dl] \\ & Y & \end{tikzcd} \quad \text{with} \begin{cases} h_{|X \times 0} = s \circ f \\ h_{|X \times 1} = \mathrm{id}_X \end{cases}$$

proof: <u>1):</u> b) $\Rightarrow$ a) is clear.

<u>a) $\Rightarrow$ b):</u> Let $S = {}^{\boxslash} g$. We know that $(\partial\Delta^n \hookrightarrow \Delta^n) \subseteq S$ by a) and we want to prove that $(\text{monos}) \subseteq S$.

We know by $\begin{cases} \text{Prop 5 that } S \text{ is saturated.} \\ \text{Prop 9 that } (\text{monos}) = \overline{(\partial\Delta^n \hookrightarrow \Delta^n)}. \end{cases}$

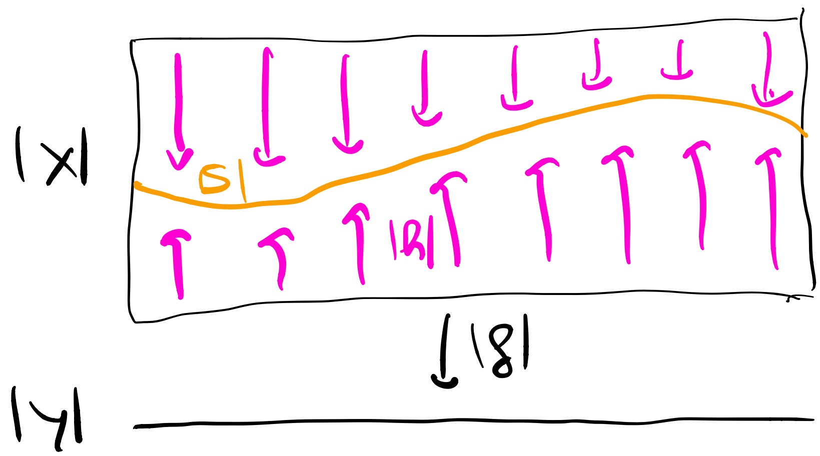
and we are done.

2) s (resp. k) is solution of:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, "\underline{\text{mono}}"'] & X \arrow[d, "g"] \\ Y \arrow[r, equal, "id"] \arrow[ur, dashed, "s"] & Y \end{tikzcd} \quad \text{resp.} \quad \begin{tikzcd} \partial\Delta^n \times X \arrow[r, "(s\circ g, id)"] \arrow[d] & X \arrow[d, "g"] \\ \Delta^n \times X \arrow[r] \arrow[ur, dashed, "k"] & Y \end{tikzcd}$$

which exist by part 1). $\square$

Rmk: Part 2) means that, not only are the fibers of a trivial fibration contractible Kan complexes, but they can be contracted "simultaneously": after geometric realisation:



Our next objective is to get similar description of inner/left/right anodyne maps as saturated classes.

Prop 11: (Small object argument)

Let S be a set of morphisms in sSet. Then every morphism δ in sSet admits a factorisation $\delta = pj$ with $j \in \overline{S}$ and $p \in S^{\boxslash}$.

Moreover this factorisation can be made functorial in f.

proof: • I give the proof under a stronger assumption: $S = \{ A_i \xrightarrow{m_i} B_i \}_{i \in I}$ with A_i has finitely non-degenerate simplices.
This assumption is satisfied in our applications and means that we can replace transfinite compositions with ω-indexed ones.

• We produce a first factorisation of $f : X \to Y$,

$$X \xrightarrow{Lf} Ef \xrightarrow{Rf} Y \quad \text{with } Lf \in \overline{S}$$

but not yet Rf in $S^{\boxslash}$.

Consider the set:

$$[S, f] = \left\{ (s_i, u, v) \,\middle|\, s_i \in S,\ fu = vs_i \right\} = \left\{ \begin{array}{ccc} A_i & \xrightarrow{u} & X \\ {\scriptstyle s_i}\downarrow & & \downarrow{\scriptstyle f} \\ B_i & \xrightarrow{v} & Y \end{array} \right\}$$

We define Ef, Lf, Rf via the diagram:

$$
\begin{tikzcd}
\coprod\limits_{(s_i,u,v)\in[S,f]} A_i \arrow[rr, "(u)"] \arrow[d, "\coprod m_i"', "\in \overline{S}"] & & X \arrow[d, "Lf"'] \arrow[dr, "f"] & \\
\coprod\limits_{(s_i,u,v)\in[S,f]} B_i \arrow[rr] \arrow[rrr, bend right, "(v)"'] & & Ef \arrow[ul, phantom, very near start, "\ulcorner"] \arrow[r, "Rf"] & Y
\end{tikzcd}
$$

By construction / Lemma 6, $Lf \in \overline{S}$.

- We iterate the construction, applying it to Rf..

$$X \xrightarrow{\in \overline{S}} Ef \xrightarrow{\in \overline{S}} E^2 f \xrightarrow{\in \overline{S}} \cdots \longrightarrow Y$$

Define $E^\infty f := \operatorname*{colim}_{n\in\mathbb{N}} E^n f$ to get a factorisation $X \xrightarrow[j]{} E^\infty f \xrightarrow[p]{} Y$ of f.

Again by construction, we have $j \in \overline{S}$,

so all that remains is to show $p \in S^{\boxslash}$.

Consider a map $A_j \to B_j$ in S and a lifting diagram
$$\begin{tikzcd} A_j \arrow[r] \arrow[d] & E^\infty f \arrow[d] \\ B_j \arrow[r] & Y \end{tikzcd}$$

- By the assumption on A_j, as we have seen in the proof of Prop II.7, A_j is <u>compact</u> in sSet: the functor

$$sSet(A_i, -) : sSet \longrightarrow Set$$

commutes with filtered colimits.

In particular,
$$\operatorname*{colim}_{n \in \mathbb{N}} sSet(A_j, E^n f) \xrightarrow{\sim} sSet(A_j, E^\infty f)$$

is a bijection. So we get $n \in \mathbb{N}$ and:

$$\begin{tikzcd} A_j \arrow[r] \arrow[d] \arrow[drr, phantom, "*"] & E^n f \arrow[r] \arrow[dr] & E^\infty f \arrow[d] \\ B_j \arrow[rr] & & Y \end{tikzcd}$$

So $\circledast \in [S, E^n f \to Y]$ and by definition of $E^{n+1} f$, there is a factorisation

$$
\begin{tikzcd}
A_j \arrow[r] \arrow[d] & E^n f \arrow[r] \arrow[d] & E^\infty f \arrow[d] \\
B_j \arrow[r] \arrow[urr, dashed] & E^{n+1} f \arrow[r] \arrow[ur] & Y
\end{tikzcd}
$$

and we are done. $\square$

Rmk: As you can see, this argument is quite formal and does not use much about sSet. It is very useful in the context of constructing model categories, which is what we are doing piece by piece (working towards the so-called Joyal model structure on sSet).

- The small object argument relies on compactness. and ultimately the behaviour of Set wrt filtered colimits. This is why saturated are much more manageable than "cosaturated" ones.

Cor 12: Let S be any set of maps in sSet.
Then $\overline{S} = {}^{\boxslash}(S^{\boxslash})$.

proof: We know by Lemma 2 and Proposition 7 that

$$\left\{\begin{array}{l} S \subseteq {}^{\boxslash}(S^{\boxslash}) \\ {}^{\boxslash}(S^{\boxslash}) \text{ is saturated} \end{array}\right. \implies \overline{S} \subseteq {}^{\boxslash}(S^{\boxslash}).$$

Conversely, let $f \boxslash S^{\boxslash}$ and apply Prop 11 to get $f = pj$ with $j \in \overline{S}$ and $p \in S^{\boxslash}$.

By $f \boxslash p$, there is a lift in

$$\begin{tikzcd} \bullet \arrow[r, "j"] \arrow[d, "f"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, equal] \arrow[ur, dashed, "s"] & \bullet \end{tikzcd}$$

Hence:

$$\begin{tikzcd} \bullet \arrow[r, equal] \arrow[d, "j"'] & \bullet \arrow[r, equal] \arrow[d, "f"] & \bullet \arrow[d, "j"] \\ \bullet \arrow[r, "p"] & \bullet \arrow[r, "s"] & \bullet \end{tikzcd}$$

"retract trick"

shows that f is a retract of j, hence in $\bar{S}$. $\square$

def 13: A *weak factorisation system* in a category C is a pair (L, R) of classes of maps such that:

- every morphism f factors as
  $$f = r\ell \quad \text{with } r \in R \text{ and } \ell \in L.$$
- $L = {}^{\boxslash}R$ and $R = L^{\boxslash}$.

A weak factorisation where the factorisation is unique is an *orthogonal factorisation system*.

- The small object argument tells you that $(\bar{S}, S^{\boxslash})$ is a weak factorisation system:
  - small object argument $\Rightarrow$ factorisation
  - $\bar{S} = {}^{\boxslash}(S^{\boxslash})$ by cor.
  - $(\bar{S})^{\boxslash} \subseteq S^{\boxslash}$ because $S \subseteq \bar{S}$

    $S^{\boxslash} \subseteq \bar{S}^{\boxslash}$ because left complements are saturated.

Let's apply this to our anodyne classes.

Cor 14: We have

$$(\text{inner anodyne}) = \overline{(\Lambda^n_k \hookrightarrow \Delta^n, 0<k<n)}$$

$$(\text{left anodyne}) = \overline{(\Lambda^n_k \hookrightarrow \Delta^n, 0\leq k<n)}$$

$$(\text{right anodyne}) = \overline{(\Lambda^n_k \hookrightarrow \Delta^n, 0<k\leq n)}$$

$$(\text{anodyne}) = \overline{(\Lambda^n_k \hookrightarrow \Delta^n, 0\leq k\leq n)}$$

Furthermore, we get weak factorisation systems:

(inner anodyne, inner fibration)

(left ——, left ——)

(right —, right —)

(anodyne, Kan fibrations)

(monos, trivial Kan fibrations)

proof: Apply corollary 12 to each class of horn inclusions. $\square$

Let us discuss some concrete examples of inner anodyne morphisms.

- For $I \subseteq [n]$, we have the I-horn:

$$\Lambda^n_I = \bigcup_{i \notin I} \Delta^{[n] \setminus i} \hookrightarrow \Delta^n$$

Lemma 15: a) Let $\emptyset \neq I \subseteq [n]$. Then

$\Lambda^n_I \hookrightarrow \Delta^n$ is:

- anodyne if $I \neq [n]$
- left anodyne if $n \notin I$
- right anodyne if $0 \notin I$
- inner anodyne, if I is not the complement of an interval. $\underset{0}{[} \quad \underset{a}{]}\text{-----}\underset{b}{[} \quad \underset{n}{]}$ $\quad a < b < n$

b) The map $I^n \hookrightarrow \Delta^n$ (spine inclusion) is inner anodyne.

proof: a) Let $\emptyset \neq I \ni i$ and $I' := I \setminus \{i\}$.

There is a pushout diagram:

$$(\square) \qquad \begin{tikzcd} \Delta^{[n]\setminus i} \cap \Lambda^n_I \arrow[r] \arrow[d] & \Delta^{[n]\setminus i} \arrow[d] \\ \Lambda^n_I \arrow[r] & \Lambda^n_{I'} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r] & \Delta^n \end{tikzcd}$$

The top horizontal map is $\simeq$ to the map $\Lambda^{[n]\setminus i}_{I'} \hookrightarrow \Delta^{[n]\setminus i}$.

Using this, it is easy to show a) by induction on $|I|$ (choosing i carefully in the inner anodyne case).

b) $I^n \hookrightarrow \Delta^n$ can be factored as

$$I^n \xhookrightarrow{f_n} \Delta^{[n]\setminus 0} \cup I^n \xhookrightarrow{g_n} \Delta^n.$$

We show by induction on n that both f_n and g_n are inner anodyne.

It is easy to check for $n \leq 2$.

For $n \geq 3$, we have a pushout diagram

$$\begin{tikzcd}
I^{[n]\setminus 0} \arrow[r] \arrow[d] & \Delta^{[n]\setminus 0} \arrow[d] \\
I^n \arrow[r, "f_n"] & \Delta^{[n]\setminus 0} \cup I^n \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

So f_n is a pushout of a smaller spine inclusion.

$\Rightarrow f_n$ inn. anodyne. Consider the pushout diagram:

$$\begin{tikzcd}
\Delta^{[n]\setminus\{0,n\}} \cup I^{[n]\setminus n} \arrow[r, "g_{n-1}"] \arrow[d] & \Delta^{[n]\setminus n} \arrow[d] & \\
\Delta^{[n]\setminus 0} \cup I^n \arrow[r] \arrow[rr, bend right, "g_n"'] & \Delta^{[n]\setminus 0} \cup \Delta^{[n]\setminus n} \arrow[r, "\circledast"] \arrow[ul, phantom, "\ulcorner", very near start] & \Delta^n
\end{tikzcd}$$

$\circledast$ is $\Lambda^n_{[n]\setminus\{0,n\}} \hookrightarrow \Delta^n$ which is inner anodyne

by part a). By induction, g_n is inner anodyne. $\square$

cor 16: Let C be an ∞-category. Then we have

$$\begin{tikzcd} I^n \arrow[r] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

contrast with

$$\begin{tikzcd} I^n \arrow[r] \arrow[d, hook] & NC \\ \Delta^n \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

« Infinity-categories admit compositions. »

---

2) Pushout-products and pullback-homs

Our goal is to prove that $\mathrm{Fun}(K, C)$ is an ∞-category whenever is, and also to prove a beautiful alternative characterisation of ∞-categories.

def 17: Let C be a cartesian closed category which admits pushouts / pullbacks (eg $C = sSet$).

Let $f : X \longrightarrow Y$ and $i : A \longrightarrow B$ two morphisms.

We define the pushout-product of f and i by

$$f \boxtimes i : (A \times Y) \amalg_{A \times X} (B \times X) \longrightarrow B \times Y$$

through the diagram:

$$\begin{tikzcd}
A \times X \arrow[r, "f \times id"] \arrow[d, "id \times i"'] & B \times X \arrow[d] \arrow[ddr, bend left, "id \times i"] & \\
A \times Y \arrow[r] \arrow[drr, bend right, "f \times id"'] & (A \times Y) \amalg_{A \times X} (B \times X) \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "f \boxtimes i"] & \\
& & B \times Y
\end{tikzcd}$$

- Dually, we define the pullback-hom

$$\langle f, i \rangle : \underline{\mathrm{Hom}}(B, X) \longrightarrow \underline{\mathrm{Hom}}(A, X) \times_{\underline{\mathrm{Hom}}(A, Y)} \underline{\mathrm{Hom}}(B, Y)$$

through the diagram:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(B, X) \arrow[r, dashed, "\langle f, i \rangle"'] \arrow[rr, bend left, "i^*"] \arrow[dr, bend right, "f_*"'] & \underline{\mathrm{Hom}}(A, X) \times_{\underline{\mathrm{Hom}}(A, Y)} \underline{\mathrm{Hom}}(B, Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Hom}}(A, X) \arrow[d, "f_*"] \\
& \underline{\mathrm{Hom}}(B, Y) \arrow[r, "i^*"] & \underline{\mathrm{Hom}}(A, Y)
\end{tikzcd}$$ $\square$

Rmk: • $\boxtimes$ is symmetric, unlike $\langle \, , \rangle$.

• The pullback-hom can be thought of as an "enriched" version of a lifting problem; for instance in sSet, we have

$$i \boxslash f \iff \langle f, i \rangle \text{ is surjective on vertices.}$$

"The target of $\langle f, i\rangle$ parametrizes families of commutative squares
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$
while the source parametrizes families of liftings
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$
".

Lemma 18:

a) Let $F : C \rightleftarrows D : G$ be an adjunction. Then we have an equivalence of lifting problems:

$$\begin{tikzcd} FA \arrow[r] \arrow[d] & A' \arrow[d] \\ FB \arrow[r] \arrow[ur, dashed, "?"] & B' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r] \arrow[d] & GA' \arrow[d] \\ B \arrow[r] & GB' \end{tikzcd}$$

b) With the notations of def 17, if we have

morphisms $\begin{cases} f : X \to Y \\ i : A \to B \\ g : S \to T \end{cases}$ in C

we have an equivalence of lifting problems

$$\begin{tikzcd} S \arrow[r] \arrow[d, "g"'] & \underline{\mathrm{Hom}}(B,X) \arrow[d, "\langle f, i \rangle"] \\ T \arrow[r] \arrow[ur, dashed, "?"] & \underline{\mathrm{Hom}}(A,X) \times_{\underline{\mathrm{Hom}}(A,Y)} \underline{\mathrm{Hom}}(B,Y) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (A \times T) \coprod_{A \times S} (B \times S) \arrow[r] \arrow[d, "i \boxtimes g"] & X \arrow[d, "f"] \\ B \times T \arrow[r] & Y \end{tikzcd}$$

In other words, $g \boxslash \langle f, i \rangle \iff (i \boxtimes g) \boxslash f$.

proof: • a) is a direct application of the natural isomorphism $D(F(-), -) \simeq C(-, G(-))$.

b) follows from a) applied to the adjunction

$B \times - \dashv \underline{\mathrm{Hom}}(B, -)$ and the

definition of the maps $\langle f, i \rangle$ and $i \boxtimes g$.

$\square$

Here is the technical heart of this

Section:

Prop 19: Let i, g be monomorphisms

in sSet. Assume that one of i or g

is inner anodyne (resp. left anodyne,

right anodyne, anodyne).

Then so is $i \boxtimes g$. $\square$

The proof is quite involved. We

start with:

Lemma 20: Let S, T be two sets of maps in sSet. Then

$$\overline{S} \boxtimes \overline{T} \subseteq \overline{S \boxtimes T}$$

proof: Let $F = (S \boxtimes T)^{\boxslash}$.

By the small object argument, we have $\overline{S \boxtimes T} = {}^{\boxslash}F$, so we have to prove $(\overline{S} \boxtimes \overline{T}) \boxslash F$. We first show $(\overline{S} \boxtimes T) \boxslash F$. Consider

$$\mathcal{A} = \{ a \mid (a \boxtimes T) \boxslash F \}$$

By Lemma 18.b), we have

$$\mathcal{A} = \{ a \mid a \boxslash \langle F, T \rangle \}$$

So $\mathcal{A}$ is a left-complement, and in particular is saturated. by Prop 7.

$S \subseteq \mathcal{A}$, hence $\overline{S} \subseteq \mathcal{A}$ and we have

$$(\overline{S} \boxtimes T) \boxslash F.$$

A similar argument with

$$\mathcal{B} = \{ b \mid (\overline{S} \boxtimes b) \boxslash F \}$$
$$= \{ b \mid b \boxslash \langle F, \overline{S} \rangle \}$$

shows that $(\overline{S} \boxtimes \overline{T}) \boxslash F$. $\square$

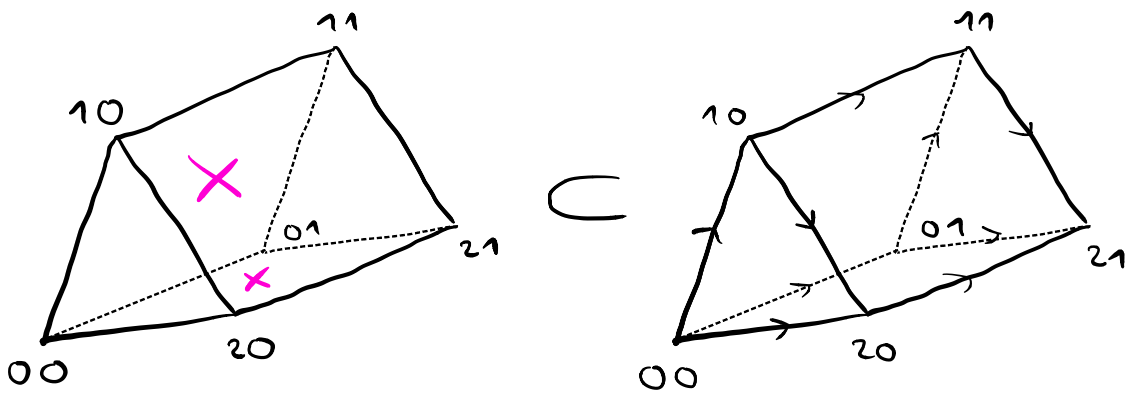
Lemma 21: We have

$$(\Lambda^n_k \hookrightarrow \Delta^n \mid 0 < k < n) \boxtimes (\partial \Delta^n \hookrightarrow \Delta^n)$$
$$\cap\!|$$
$$(\text{inner anodyne maps})$$

Proof: This is difficult combinatorics!

Following Rezk, I will just explain a special case: $(\Lambda^2_1 \subset \Delta^2) \boxtimes (\partial\Delta^1 \subset \Delta^1)$.

$$\underbrace{(\Lambda^2_1 \times \Delta^1) \amalg (\Delta^2 \times \partial\Delta^1)}_{\Lambda^2_1 \times \partial\Delta^1} \subset \Delta^2 \times \Delta^1$$



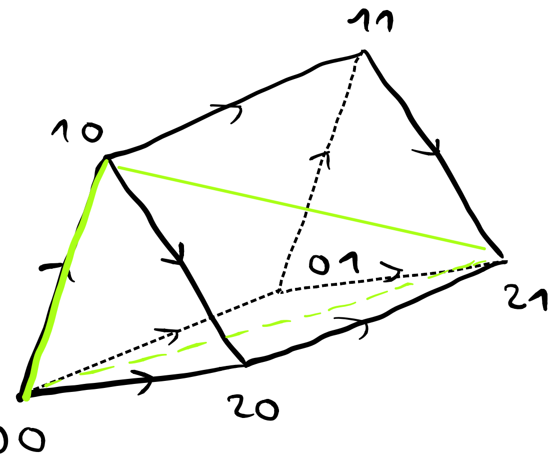
« tent » $\subset$ prism.

We attach the following simplices in order:

1. $\langle 00, 10, 21 \rangle$
2. $\langle 00, 10, 20, 21 \rangle$
3. $\langle 00, 10, 11, 21 \rangle$
4. $\langle 00, 01, 11, 21 \rangle$

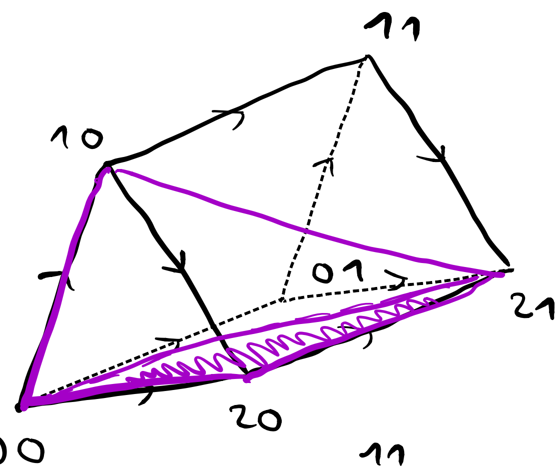
In each case, this is a pushout along an inner horn. This can be seen geometrically with a little patience!

1



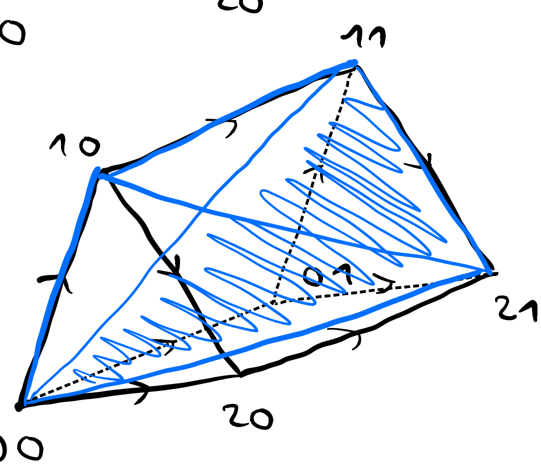
is attaching $\Lambda^2_1 \hookrightarrow \Delta^2$

2



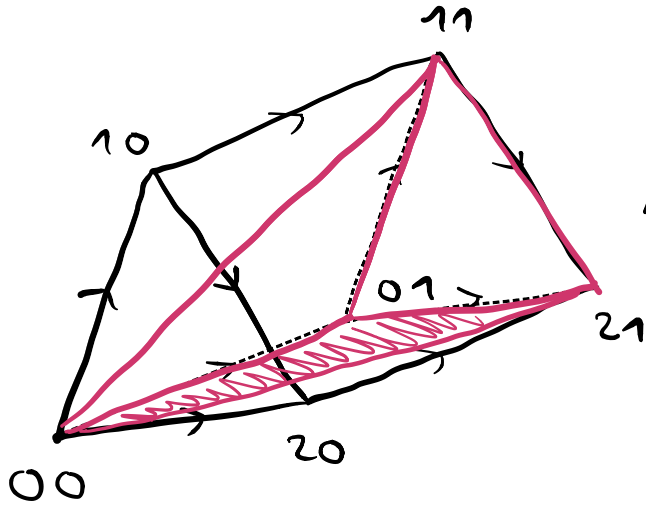
" $\Lambda^3_1 \hookrightarrow \Delta^3$

3



" $\Lambda^3_1 \hookrightarrow \Delta^3$

4



$\Lambda^3_2 \hookrightarrow \Delta^3$

$\square$

proof of 19:

I only explain the case of inner anodyne maps. The others use variants of Lemma 20 which I don't want to get into; moreover for the application in this section, this is the case we need.

We know that

$$(\text{monos}) = \overline{(\partial\Delta^n \hookrightarrow \Delta^n)}$$

by Prop 9.

We see that

$$(\text{inner anodyne}) \boxtimes (\text{monos})$$

$$= \overline{\left(\Lambda^n_k \hookrightarrow \Delta^n \mid 0<k<n\right)} \boxtimes \overline{\left(\partial\Delta^n \hookrightarrow \Delta^n\right)}$$

Lemma 20

$$\subseteq \overline{\left(\Lambda^n_k \hookrightarrow \Delta^n \mid 0<k<n\right) \boxtimes \left(\partial\Delta^n \hookrightarrow \Delta^n\right)}$$

Lemma 21

$$\subseteq (\text{inner anodyne})$$

$\square$

Now the hand work is done, and we get:

Prop 22: Let $f: X \to Y$ be an (inner, left, right) fibration and $i: A \to B$ a monomorphism. Then

1) $\langle f, i \rangle$ is an (inner, left, right) fibration.

2) If i is moreover (inner, left, right) anodyne, then $\langle f, i \rangle$ is a trivial fibration.